BRIEF NOTE CREEP-STABILITY ANALYSIS OF VISCOELASTIC CYLINDRICAL SHELLS

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Communicated by Xavier J. R. Avula

(Received March 1984; revised November 1985)

Abstract—The paper presents the creep-stability analysis of viscoelastic cylindrical shells under axial compression. The mechanical properties of the material are described by the constitutive equations of the linear viscoelastic theory in terms of convolution integral operators. The approximate analytical solution to the problem is obtained by means of a modification of the quasi-elastic method. As a result, the instability condition for the shell is formulated. It is shown that for viscoelastic materials with limited creep, there is a safe load limit below which the structure is asymptotically stable. Any load above the safe load limit leads to buckling at the corresponding critical time.

1. INTRODUCTION

Creep stability of circular cylindrical shells has been studied by numerous researchers using various concepts and constitutive laws. A number of investigations utilize the initial imperfection approach followed by the conclusion that under creep conditions initial imperfections develop with time, leading eventually to collapse of the structure. The critical time in such analyses is usually defined in terms of infinite deformations or infinite deformation rates. Comprehensive reviews of these studies are given by Hoff[1, 2] and Kurshin[3].

In recent years an increasing interest has been attracted by applications of the classical bifurcation theory to the creep-stability research. In particular, creep stability of cylindrical shells is treated in many publications[4–8] as an instantaneous process which involves time-dependent constitutive terms and is characterized as branching into a different equilibrium configuration. Respectively, the critical time is associated with the instant at which bifurcation first becomes possible. The results from these studies depend upon the basic assumptions as to the creep properties of the structure.

The present paper is concerned with the stability analysis of circular cylindrical shells whose material properties are defined by the constitutive equations of the linear viscoelastic theory. Using the concept of bifurcation the exact linear eigenvalue problem is formulated in terms of two destabilizing parameters: the compressive load and time. The problem is treated by means of the quasi-elastic solution technique which utilizes the concept of a time-dependent elastic material as a model of the actual viscoelastic response. This approach to the creep-stability analysis of viscoelastic structures is discussed by the author in [9].



2. STATEMENT OF THE PROBLEM

Consider a circular cylindrical shell (Fig. 1) of length l, radius r and thickness h, whose material properties can be described by the linear theory of hereditary viscoelasticity. The shell is simply supported at both ends and is subjected to a uniform axial compression of magnitude, p, which is less than the elastic critical pressure p_e . It is assumed that the load is suddenly applied at the time, t = 0, but this does not imply rates sufficiently great to cause the excitation of a dynamic response of the structure.

The quasi-static state of the shell is governed by a set of simultaneous equations comprising the equations of equilibrium, kinematic relations and the viscoelastic constitutive law. Of all these conditions, it is only the viscoelastic constitutive equations which differ from those of the corresponding elastic problem. Other equations follow directly from the theory of elastic stability.

In this study, the neutral equilibrium of the shell is described by the Donnell equations[10], and the constitutive equations of the linear viscoelastic theory are expressed in terms of hereditary integral operators, E^* and ν^* , involving experimentally measurable creep or relaxation functions[11].

The operator E^* can be defined by the uniaxial stress-strain relation presented in either of two equivalent forms:

$$\sigma(t) = E^*\{\epsilon(t)\} = E(1 - R^*)\{\epsilon(t)\}$$
 (1)

or

$$\epsilon(t) = \frac{1}{E^*} \{ \sigma(t) \} = \frac{1}{E} (1 + \Gamma^*) \{ \sigma(t) \},$$
 (2)

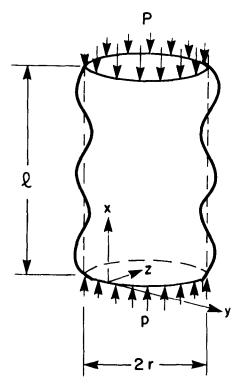


Fig. 1. Bufurcation of a viscoelastic cylindrical shell under axial compression.

where E denotes the instantaneous elastic modulus of the material, and the notations R^* and Γ^* are used to signify two integral operators of the Stjelties convolution type

$$R^*\{\epsilon(t)\} = \int_0^t R(t - \tau) \, \epsilon(\tau) \, d\tau \tag{3}$$

and

$$\Gamma^* \{ \sigma(t) \} = \int_0^t \Gamma(t - \tau) \ \sigma(\tau) \ d\tau. \tag{4}$$

The kernel functions, $R(t - \tau)$ and $\Gamma(t - \tau)$, characterize, respectively, the relaxation and the creep properties of the viscoelastic material. It follows from Eqs. (1) and (2) that the operator E^* can be expressed either in the form

$$E^* = E(1 - R^*) \tag{5}$$

or

$$E^* = E/(1 + \Gamma^*). {(6)}$$

The operator ν^* is, in general, of the same form as E^* but involves a different kernel function. In this study, however, it is assumed, for simplicity, that

$$\nu^* = \nu, \tag{7}$$

where ν denotes the instantaneous Poisson's ratio. Equation (7) implies that in a uniaxial extension test under stress-relaxation conditions, the time-dependent portion of the lateral strain can be neglected. This assumption is found to be sufficiently accurate for most actual viscoelastic materials.

In terms of the middle-surface shell forces, N_x , N_y and N_{xy} , the viscoelastic constitutive equations are defined in the form

$$N_{x} = \sigma_{xx}h = \frac{h}{1 - \nu^{2}} E^{*} \{ \epsilon_{xx} + \nu \epsilon_{yy} \},$$

$$N_{y} = \sigma_{yy}h = \frac{h}{1 - \nu^{2}} E^{*} \{ \epsilon_{yy} + \nu \epsilon_{xx} \},$$

$$N_{xy} = \tau_{xy}h = \frac{h}{2(1 + \nu)} E^{*} \{ \gamma_{xy} \},$$
(8)

where the strain components, ϵ_{xx} , ϵ_{xy} and γ_{xy} , are functions of the coordinates, x and y, and the time, t.

These equations, solved simultaneously with the kinematic relations and the equations of equilibrium of the cylindrical shell, reduce the viscoelastic stability problem to a single linear homogeneous integro-differential equation

$$D^*\{\nabla^8 w\} + \frac{h}{r^2} E^* \left\{ \frac{\partial^4 w}{\partial x^4} \right\} + ph \nabla^4 \frac{\partial^2 w}{\partial x^2} = 0, \tag{9}$$

in which ∇^4 and ∇^8 indicate two and four successive applications of the Laplace operator;

w = w(x, y, t) denotes the lateral displacement of the shell, and D^* is defined as

$$D^* = \frac{h^3}{12(1-\nu^2)}E^*. ag{10}$$

Note that Eq. (9) is of the same form as the governing equation of the associated elastic stability problem in which the elastic modulus, E, is replaced by the integral operator, E^* .

For a simply supported cylindrical shell

$$w = \frac{\partial^2 w}{\partial x^2} = 0, \quad \text{at} \quad x = 0, l.$$
 (11)

These boundary conditions are stationary in time.

The initial condition to the problem is associated with the assumption that the response of the structure at the time of the load application is instantaneously elastic. One notes that this condition is satisfied automatically since at t=0 the viscoelastic constitutive equations coincide with the corresponding equations for an elastic material with the instantaneous Young's modulus, E.

3. METHOD OF SOLUTION

In this paper the problem under consideration is treated by means of the quasi-elastic method suggested by Schapery[12] for the linear viscoelastic stress analysis. The method is based upon the fact that the behaviour of many engineering materials with fading memory can be approximately replaced by the response of a fictitious elastic material the mechanical properties of which depend parametrically on time. Similarly to the Laplace transform technique, the quasi-elastic method utilizes the associated elastic solution but does not involve complications of the inverse transformation procedure.

Application of the quasi-elastic method to the uniaxial stress-strain relation given by Eq. (1) results in the approximation

$$\sigma(t) \simeq E(t) \cdot \epsilon(t), \tag{12}$$

where

$$E(t) = E^*\{1\} = E[1 - R^*\{1\}]. \tag{13}$$

One can observe that Eq. (12) is derived from Eq. (1) by replacing the action of the integral operator $E^*\{\epsilon(t)\}$ by the product $E(t) \cdot \epsilon(t)$. Equation (12) is identical in form with the elastic stress-strain relation in which E(t) represents a time-dependent Young's modulus. Note that, according to Eq. (13), the function E(t) is associated with the relaxation properties of the actual viscoelastic material.

An alternative expression for the function E(t) can be derived from Eq. (2) in the form

$$E(t) = E/[1 + \psi(t)],$$
 (14)

where ψ denotes the creep function of the material

$$\psi(t) = \Gamma^* \{1\} = \int_0^t \Gamma(t - \tau) d\tau.$$
 (15)

It is shown in [12] that the quasi-elastic solution technique is equally applicable to twoand three-dimensional viscoelastic equations, and that the method provides sufficiently accurate results when the characteristic parameters involved in the problem are slowly varying functions of time. Accuracy of the quasi-elastic method in application to the creepstability analysis of viscoelastic structures is discussed in [9].

The quasi-elastic approximation of the stress-strain relations given by Eqs. (8) is of the form

$$N_{x} \simeq \frac{hE(t)}{1 - \nu^{2}} (\epsilon_{xx} + \nu \epsilon_{yy}),$$

$$N_{y} \simeq \frac{hE(t)}{1 - \nu^{2}} (\epsilon_{yy} + \nu \epsilon_{xx}),$$

$$N_{xy} \simeq \frac{hE(t)}{2(1 + \nu)} \gamma_{xy},$$
(16)

where E(t) is defined by either Eq. (13) or (14).

Equations (16), solved simultaneously with the equations of equilibrium of the shell and the kinematic relations, result in the differential equation

$$D(t) \nabla^4 w_0 + \frac{h}{r^2} E(t) \frac{\partial^4 w_0}{\partial x^4} + ph \nabla^4 \frac{\partial^2 w_0}{\partial x^2} = 0$$
 (17)

in which time t is a parameter, w_0 denotes the quasi-elastic approximation of the lateral deflection w, $w_0 \approx w$, and D(t) is defined as

$$D(t) = \frac{h^3 E(t)}{12(1 - \nu^2)} \,. \tag{18}$$

Equation (17) can be treated as the governing equation of the elastic-stability problem in which a fictitious cylindrical shell, geometrically similar to the original viscoelastic shell, is acted upon by the same axial compressive load and has elastic properties characterized by a time-dependent elastic modulus, E(t). Note that the function E(t) depends upon the properties of the actual viscoelastic material and can be derived either from Eq. (13) or (14), provided that the creep or relaxation characteristics of this material are specified.

Equation (17) depends parametrically on time. By assigning a certain value of t, one arrives at the governing equation of the corresponding elastic-stability problem in which the critical load, p_{cr} , can be derived using the regular solution procedure of the theory of elastic stability. Different values of the parameter t generate, respectively, a sequence of corresponding magnitudes of the critical load so that a continuous function of time, $p_{cr}(t)$, can be derived in the form

$$\frac{p_{cr}(t)}{p_e} = \frac{1}{1 + \psi(t)},\tag{19}$$

where p_e denotes the elastic critical load defined by

$$p_e = \frac{Eh}{r} \frac{1}{\sqrt{3(1 - v^2)}} \,. \tag{20}$$

4. DISCUSSION

Equation (19) represents the condition of instability of a viscoelastic cylindrical shell under axial compression. This condition involves two destabilizing parameters: the applied load p, and the time t. It is of particular interest that the ratio p_{cr}/p_e depends only upon the creep properties of the viscoelastic material.

Further in the analysis, two types of linear viscoelastic materials will be distinguished: those exhibiting limited and unlimited creep. The first type comprises the materials defined in [13] as viscoelastic solids and is characterized by a creep function, $\psi(t)$, which tends to a limiting constant value, ψ_{∞} , as $t \to \infty$. The second type includes so-called linear viscoelastic fluids with the creep function unlimited in time, $\psi_{\infty} = \infty$.

It follows from Eq. (19) that for linear viscoelastic solids there is a particular magnitude of the compressive load, P_s , below which the cylindrical shell remains asymptotically stable. This load can be defined as the safe load limit, P_s , which is given by

$$\frac{P_s}{P_e} = \frac{1}{1 + \psi_\infty} \,. \tag{21}$$

Clearly, for linear viscoelastic fluids, $P_s = 0$.

Any compressive load above P_s leads eventually to bifurcation of equilibrium which occurs at the critical time, t_{cr} . For a compressive load within the limits $P_s \leq P \leq P_e$ the corresponding value of t_{cr} can be derived using Eq. (19), provided that the creep function of the material, $\psi(t)$, is specified.

The diagram shown in Fig. 2 presents a geometrical interpretation of the typical critical load-time relation defined by Eq. (19). One can observe that, for the two limiting cases, $P = P_s$ and $P = P_e$, the corresponding values of t_{cr} are defined, respectively, as $t_{cr} \to \infty$ and $t_{cr} = 0$. The latter result indicates that at a compressive load of the same magnitude as the classical elastic critical load, bifurcation of equilibrium occurs at the time of the load application.

As an illustration of the above results, two simple models are considered representing viscoelastic materials with limited and unlimited creep. The first model shown in Fig. 3(a)

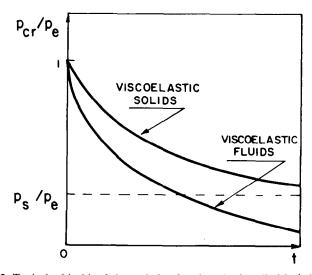


Fig. 2. Typical critical load-time relation for viscoelastic cylindrical shells.

is known as the three-parameter standard solid. It is characterized by the creep function

$$\psi(t) = \frac{E}{E_1} (1 - e^{-\mu t}), \qquad \mu = E_1/\eta_1$$
 (22)

which is limited by $\psi_{\infty} = E/E_1$. The Maxwell-Kelvin model shown in Fig. 3(b) represents a viscoelastic material with the creep function unlimited in time:

$$\psi(t) = \frac{E}{E_1} (1 - e^{-\mu t}) + \frac{E}{\eta} t.$$
 (23)

Under the assumption that $E/E_1 = 0.5$ and $\eta_1/\eta = 0.1$, one arrives at $\psi_\infty = 0.5$ and $p_s/p_e = 0.67$ in the case of the three-parameter standard solid. For the Maxwell-Kelvin material, $\psi_\infty = \infty$ and, respectively, $p_s = 0$. The load-critical time relations for both viscoelastic material models are presented in Fig. 4.

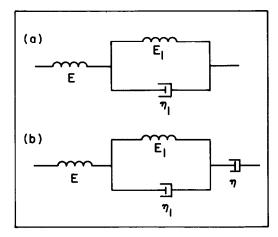


Fig. 3. Simple viscoelastic material models: (a) three-parameter standard solid; (b) Maxwell-Kelvin material.

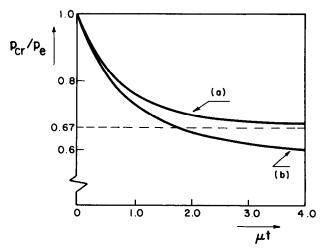


Fig. 4. The critical load-time relation: (a) three-parameter standard solid; (b) Maxwell-Kelvin material.

5. CONCLUDING REMARKS

The creep-stability problem for linearly viscoelastic cylindrical shells is formulated as an eigenvalue problem of a linear homogeneous integro-differential equation subject to certain boundary and initial conditions. The governing equation is solved approximately using the quasi-elastic solution technique. The method implies that the behaviour of a linearly viscoelastic material can be represented by an equivalent time-dependent elastic material model. Following from this approach, the instability condition for viscoelastic cylindrical shells under axial compression is obtained in a concise analytical form. It is shown that for viscoelastic materials with limited creep there is a particular value of the compressive load, referred to as the safe load limit, below which the initial equilibrium of the shell is asymptotically stable. For any load above the safe load limit, the corresponding critical time can be derived from the instability condition, provided that the creep function of the material is specified.

One notes that the critical load-time relation given in the form of Eq. (19) coincides qualitatively with the results derived in [8] and [14] by means of different solution procedures.

It is of interest that using the Laplace-transformation technique in the creep-stability analysis of viscoelastic columns with limited creep, Distefano[15] arrived at a so-called "viscoelastic critical load" which is of the same magnitude as the safe load limit, p_s , given by Eq. (21). Identical equations for p_s have been derived in [9] and [16] for linear viscoelastic spherical shells and circular arches. It appears from the latter observation that, in general, for viscoelastic thin-walled structures the relation of the safe load limit to the corresponding elastic critical load depends only upon the long-term creep properties of the linear viscoelastic material. This conclusion, however, requires further investigations.

Acknowledgement—This research was sponsored by the Natural Sciences and Engineering Research Council of Canada through Grant No. U-0041.

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